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PATHWAYS AND BARRIERS TO COUNTING

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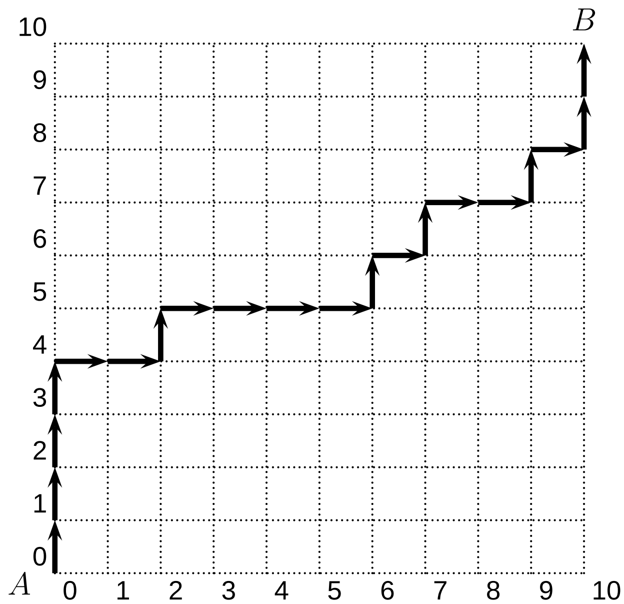
In this article I want to present some simple ideas for obtaining interesting identities related to the binomial co-efficients $\binom{n}{r}$ which turn up in the 3-Unit course.

Unordered Selections

It is well known that the number of ways to choose r distinct objects from n distinct objects, where the order of the selection is unimportant is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$



Another notation that is in common use is to write ${}^nC_r = \binom{n}{r}$, so ${}^{40}C_6 = \binom{40}{6}$.

I now want to look at some interesting examples which use $\binom{n}{r}$ and derive some nice properties that these numbers have.

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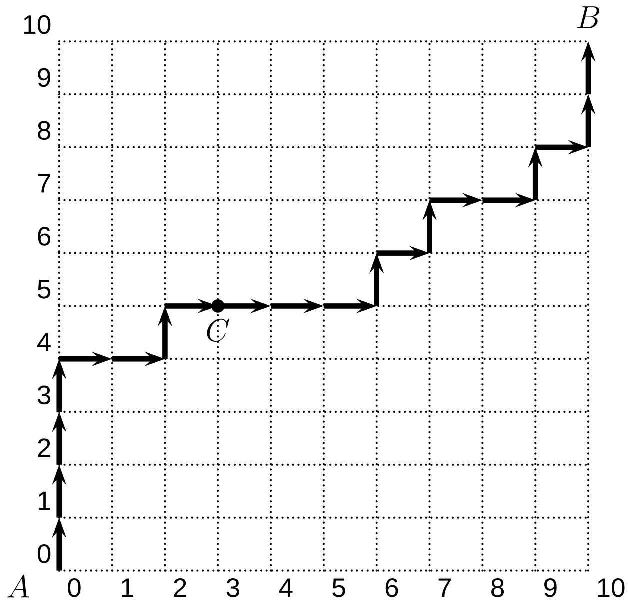
On the grid shown above, we wish to move from A to B such that we must either move to the right or upwards. How many paths are there from A to B?

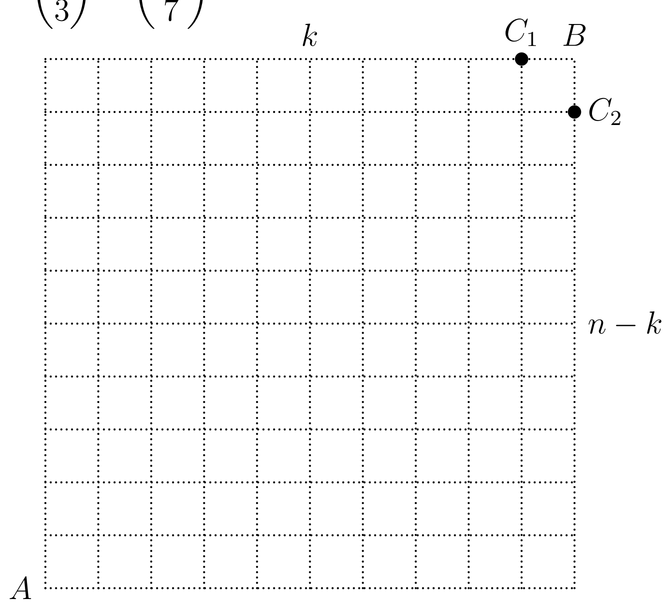
Here is one possible path $UUUURRURRRRURURRURUU$, where U denotes one move up, and R denotes one move to the right.

Observe that there will always be 20 moves from A to B, 10 up and 10 to the right. To specify a particular path, we have to say which of the twenty moves are up. Once we have decided which ones are up then the rest must be to the right. There are therefore $\binom{20}{10}$ possible paths.



How many paths pass through the point C with co-ordinates $(3, 5)$?

To get from A to C we have to choose 3 moves to the right from a total of 8 moves and from C to B we have to choose 7 moves to the right from a total of 12 moves. The answer then is $\binom{8}{3} \times \binom{12}{7}$.



We can use this idea to prove some standard identities for $\binom{n}{r}$ combinatorially.

If we have a $k \times (n-k)$ grid, (where $n \geq k$), we count the number of ways to get from A to B. There are n moves in total and we have to choose k of these moves towards the right or $n-k$ moves in the upwards direction. Hence

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n \\ n-k \end{array} \right).$$

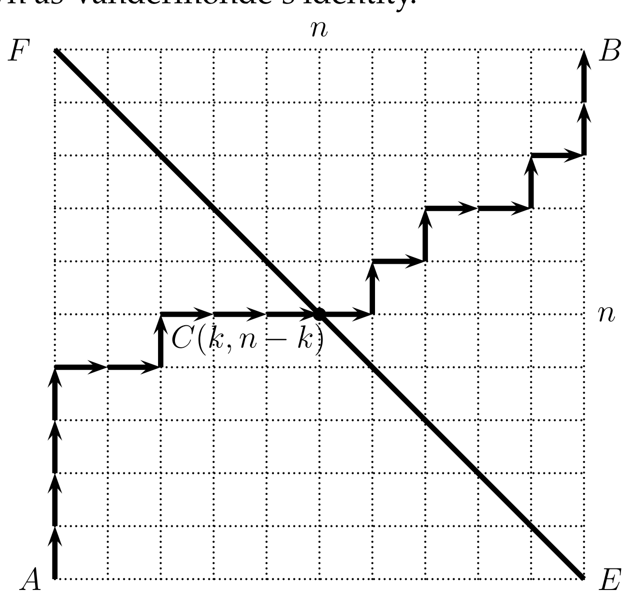
We can also prove that

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \left(\begin{array}{c} n-1 \\ k \end{array} \right) + \left(\begin{array}{c} n-1 \\ k-1 \end{array} \right).$$

Consider the grid with $n-k$ upward steps and k steps to the right. Any path from A to B must pass either through the point C_1 or C_2. Once we reach either of these points there is only one way to finish. There are $\binom{n}{k}$ ways to get from A to B while there are $\binom{n-1}{k-1}$ ways to reach C_1 and $\binom{n-1}{k}$ ways to reach C_2. Hence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This is known as Vandermonde's identity.



Furthermore, on the $n \times n$ board, every path must pass through exactly one of the points on the diagonal EF. There are $\binom{2n}{n}$ paths from A to B and if we take a general point $C(k, n-k)$ on the diagonal, there are $\binom{n}{k}\binom{n}{n-k}$ paths from A to B passing through C. Hence

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$

Using the basic fact that $\binom{n}{k} = \binom{n}{n-k}$, we can write this result as

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Another Counting Problem

Find the number of solutions in the non-negative integers to

$$x_1 + x_2 + x_3 + x_4 = 6$$

For example, $3+1+2+0$ is a solution, and so is $2+2+1+1$.

At first glance this is a fairly difficult problem. To solve it, consider a string of 9 crosses (you will see where the 9 comes from soon).

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Choose three crosses and replace them with barriers, for example

$$\times \quad \times \quad | \quad \times \quad | \quad \times \quad \times \quad | \quad \times$$

If we add the number of crosses between the barriers we have

$$2|1|2|1$$

which corresponds to a solution of the equation. Conversely, every solution to the equation will correspond to six crosses separated by 3 barriers. The number of solutions then will be the number of ways to choose 3 barriers from the 9 crosses, i.e. $\binom{9}{3}$.

To solve the equation

$$x_1 + x_2 + x_3 + \cdots + x_{10} = 23$$

we will need to draw $23 + 9 = 32$ crosses and choose 9 of them to be barriers. The number of solutions in the non-negative integers will then be $\binom{32}{9}$ and you can easily generalise the formula.

Notice that we allowed the possibility for a solution with one or more of the variables equal to 0. Suppose we seek a solution to

$$x_1 + x_2 + x_3 + \cdots + x_{10} = 23$$

in the positive integers. We could imagine the 10 variables as 10 urns into which we have to place 23 marbles. To ensure that no urn is empty we could put one marble per urn, leaving 13 marbles to then distribute among the urns. The above method says that this can be done in $\binom{22}{9}$ ways.

In general, the number of solutions to

$$x_1 + x_2 + \cdots + x_m = n$$

in the non-negative integers is $\binom{n+m-1}{m-1}$ and in the positive integers it will be $\binom{n-1}{m-1}$.

Exercise How many solutions are there to

$$x_1 + x_2 + x_3 + \cdots + x_{10} = 24$$

in the non-negative **even** integers?

A Final Example

Suppose we were to expand out $(x+y+z)^{10}$. How many terms would we get after simplifying?

Each term is of the form $x^i y^j z^k$, where $i + j + k = 10$ and we need i, j, k to be non-negative integers. This is precisely the problem we looked at before, so the number of terms is simply $\binom{10+3-1}{3-1} = \binom{12}{2} = 66$.

You might like to think about how we would find the co-efficient of say $x^2y^3z^5$ in the above expansion.