8824 – 9702M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2024

Mathematics: analysis and approaches

Higher level

Paper 3

25 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a) $T_2 = 1.1(6000) - 500$ OR $T_2 = 6600 - 500$ ***A2***

$= 6100$ ***AG***

[2 marks]

(b) (i) $T_3 = 1.1(6100) - 500$ $\left(T_3 = 1.1(1.1(6000) - 500) - 500\right)$ OR $T_3 = 6710 - 500$ ***A1***

$= 6210$ ***AG***

[1 mark]

(ii) $T_4 = 1.1(6210) - 500$ OR $T_4 = 6831 - 500$ ***(A1)***

$= 6331$ ***A1***

[2 marks]

continued…

Question 1 continued.

(c) (i) **EITHER**

attempts to eliminate the denominator ***(M1)***

$T_n = 6000(1.1)^{n-1} - 5000\left((1.1)^{n-1} - 1\right)$ $\left(= (6000 - 5000)(1.1)^{n-1} + 5000\right)$ ***A1***

Note: Accept correct equivalent forms, eg. $T_n = 6000(1.1)^{n-1} - 5000(1.1)^{n-1} + 5000$.

OR

attempts to express T_n with a common denominator of 0.1 ***(M1)***

$$T_n = \frac{600(1.1)^{n-1} - 500\left((1.1)^{n-1} - 1\right)}{0.1} \quad \left(= \frac{600(1.1)^{n-1} - 500(1.1)^{n-1} + 500}{0.1}\right)$$ ***A1***

Note: Accept correct equivalent forms.

THEN

$T_n = 1000(1.1)^{n-1} + 5000$ ***AG***

[2 marks]

(c) (ii) **EITHER**

attempts to find T_6 using the explicit formula for T_n ***(M1)***

$T_6 = 1000(1.1)^{6-1} + 5000$ OR $T_6 = 6610.51$

OR

attempts to find T_6 using $T_{n+1} = 1.1T_n - 500$ ***(M1)***

$T_4 = 6331$, $T_5 = 6464.1$, $T_6 = 6610.51$

OR

attempts to find T_6 using a finance solver application ***(M1)***

$\text{N} = 5,\ \text{I} = 10\%,\ \text{PV} = -6000$ and $\text{PMT} = 500$

THEN

6611 ***A1***

Note: Accept 6610.

[2 marks]

continued…

Question 1 continued.

(d) (i) **METHOD 1**

attempts to use the explicit formula for D_n to find an expression for $D_{n+1} - D_n$ ***(M1)***

$= 1.1D_n - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$

$= 1.1\left(-1500(1.1)^{n-1} + 7500\right) - 750 - \left(-1500(1.1)^{n-1} + 7500\right)$ ***A1***

EITHER

$= -1650(1.1)^{n-1} + 8250 - 750 + 1500(1.1)^{n-1} - 7500$ ***A1***

OR

$= 1500(1.1)^{n-1} - 1500(1.1)^{n}$

$= 1500(1.1)^{n-1}(1 - 1.1) \ \left(= 1500(1.1)^{n-1}(-0.1)\right)$ ***A1***

THEN

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

METHOD 2

attempts to express $D_{n+1} - D_n$ in terms of D_n ***(M1)***

$D_{n+1} - D_n = 1.1D_n - 750 - D_n$

$= 0.1D_n - 750$ ***(A1)***

$= 0.1\left(-1500(1.1)^{n-1} + 7500\right) - 750 \ \left(= -150(1.1)^{n-1} + 750 - 750\right)$ ***A1***

$D_{n+1} - D_n = -150(1.1)^{n-1}$ ***AG***

[3 marks]

continued…

Question 1 continued.

(ii) $D_{n+1} - D_n < 0$ (because $(1.1)^{n-1} > 0$ and so $-150(1.1)^{n-1} < 0$) ***R1***

Note: Award ***R1*** for responses such as:
'The difference is negative'.
$-150(1.1)^{n-1} < 0$
$D_n = D_{n+1} + 150(1.1)^{n-1}$
Accept responses such as:
$D_n - D_{n+1} > 0$ or 'the difference is positive' as long as $D_n - D_{n+1}$ is referred to.
Do not accept arguments based on specific numerical examples.

therefore $D_{n+1} < D_n \Rightarrow D_n > D_{n+1}$ ***AG***

(and hence the predicted number of trout at the start of any year will be greater than the predicted number at the start of the next year)

[1 mark]

continued…

Question 1 continued.

(e) **METHOD 1**

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $D_n \le 0$ ($D_n = 0$ OR $D_n < 0$) ***(M1)***

Note: Award ***(M1)*** for attempting to obtain $-1500(1.1)^{n-1} \le -7500$ or $-1500(1.1)^{n-1} = -7500$ (or equivalent) and then attempting to take the log of both sides. For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $D_n = 0$.

EITHER

$17.8863...\left(=\log_{1.1}5+1,\ =\dfrac{\ln 5}{\ln 1.1}+1\right)$ (critical value of n seen anywhere) ***(A2)***

OR

$D_{17} = 607.540...$ ($607.540... < 750$) ***(A2)***

OR

$D_{18} = -81.7054...(< 0)$ ***(A2)***

THEN

trout disappears from the lake during (at the end of) the 17th year ***A1***

Note: Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

Note: Award as above for $D_{n+1} \le 0$ ($D_{n+1} = 0$ OR $D_{n+1} < 0$).

continued…

Question 1 continued.

METHOD 2

$S_n = \sum (D_{n+1} - D_n)$ is the total decrease in the number of trout by the end of year n or start of year $n+1$

attempts (numerically, graphically, algebraically or by trial & improvement) to find the least value of n such that $S_n \le -6000$ ($S_n = -6000$ OR $S_n < -6000$) ***(M1)***

$$\frac{-150\left((1.1)^n - 1\right)}{1.1-1} \left(= -1500\left((1.1)^n - 1\right)\right) \le -6000$$

> **Note:** Award ***(M1)*** for attempting to obtain $-1500\left((1.1)^n - 1\right) \le -6000$ or $-1500\left((1.1)^n - 1\right) = -6000$ (or equivalent) and then attempting to take the log of both sides.
>
> For the use of trial & improvement, award ***(M1)***, for considering values of n either side of $S_n = -6000$.

$n = 16.8863...$ ***(A2)***

trout disappears from the lake during (at the end of) the 17th year ***A1***

> **Note:** Award ***A1*** for $n = 18$ OR ('during the) 18th year' or 'start of the 18th year'.

[4 marks]

(f) **METHOD 1**

$C_n = (6000 - 10d)(1.1)^{n-1} + 10d$ ***(A1)***

EITHER

$6000 - 10d = 0$ (recognizes that C_n is independent of n) ***(M1)***

OR

$10d = 6000$ (sets $C_n = 6000$ and compares coefficients) ***(M1)***

THEN

$d = 600$ ***A1***

continued…

Question 1 continued.

METHOD 2

EITHER

$C_{n+1} - C_n \left(= 0\right)$

$6000\left(1.1\right)^n - 10d\left(\left(1.1\right)^n - 1\right) - \left(6000\left(1.1\right)^{n-1} - 10d\left(\left(1.1\right)^{n-1} - 1\right)\right)\left(= 0\right)$ ***(A1)***

OR

$0.1C_n - d\left(= 0\right)$ (similar to part (d) (i) method 2)

$0.1\left(6000\left(1.1\right)^{n-1} - 10d\left(\left(1.1\right)^{n-1} - 1\right)\right) - d\left(= 0\right)$ ***(A1)***

THEN

attempts to expand to eliminate the linear terms in d and then attempts to factorize ***(M1)***

$600\left(1.1\right)^{n-1} - d\left(1.1\right)^{n-1}\left(= 0\right)$ $\left(\left(600 - d\right)\left(1.1\right)^{n-1}\left(= 0\right)\right)$

$d = 600$ ***A1***

METHOD 3

considers $C_{n+1} = 1.1C_n - d$ and $C_1 = 6000$ ***(A1)***

EITHER

$6000 = 1.1 \times 6000 - d$ $\left(C_1 = C_2 = 6000\right)$ ***(A1)***

OR

systematically varies the value of d ***(M1)***

THEN

$d = 600$ ***A1***

continued…

Question 1 continued.

METHOD 4

EITHER

considers a specific $C_n = C_{n+1}$, for example, $C_2 = C_3$ ***(A1)***

$6000(1.1) - 10d(1.1-1) = 6000(1.1)^2 - 10d\left((1.1)^2 - 1\right)$

attempts to solve their $C_n = C_{n+1}$ for d ***(M1)***

for example, $1.1d = 660$

OR

considers a specific $C_n = 6000$, for example, $C_2 = 6000$ ***(A1)***

$6000(1.1)^{2-1} - 10d\left((1.1)^{2-1} - 1\right) = 6000$

attempts to solve their $C_n = 6000$ for d ***(M1)***

$6600 - d = 6000$

THEN

$d = 600$ ***A1***

[3 marks]

continued…

Question 1 continued.

(g) $n=1$: $\text{LHS}=u_1$ and $\text{RHS}=\left(u_1r^{1-1}-\dfrac{d\left(r^{1-1}-1\right)}{r-1}\right)=u_1r^0-\dfrac{d\left(r^0-1\right)}{r-1}=u_1$ ***R1***

$\text{LHS}=\text{RHS}$ and so true for $n=1$

> **Note:** Award ***R0*** for considering $n=2$.
>
> Subsequent marks after this ***R1*** are independent of this mark and can be awarded.

assume true for $n=k$ $\left(u_k=u_1r^{k-1}-\dfrac{d\left(r^{k-1}-1\right)}{r-1}\right)$ ***M1***

> **Note:** The assumption of truth must be apparent. Do not award ***M1*** for statements such as "let $n=k$" or "assume that $n=k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

attempts to substitute for u_k in $u_{k+1}=ru_k-d$ ***M1***

$$=r\left(u_1r^{k-1}-\frac{d\left(r^{k-1}-1\right)}{r-1}\right)-d$$

$$=u_1r^k-\frac{dr\left(r^{k-1}-1\right)}{r-1}-\frac{d\left(r-1\right)}{r-1}\quad\left(=u_1r^k-\frac{dr^k-dr}{r-1}-d\right)$$ ***A1***

$$=u_1r^k-\frac{dr^k+dr-dr-d}{r-1}\quad\left(=u_1r^k-\frac{dr^k-d}{r-1}\right)$$ ***A1***

$$=u_1r^k-\frac{d\left(r^k-1\right)}{r-1}$$ ***A1***

since true for $n=1$ and true for $n=k+1$ if true for $n=k$, hence true for all $n\left(\in\mathbb{Z}^+\right)$ ***R1***

> **Note:** To obtain the final ***R1***, four of the previous six marks must have been awarded.

[7 marks]

Total [27 marks]

2. (a) (i) attempts to solve ***(M1)***

EITHER

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

OR

$$(x+2)^2 - 3(=0)$$

OR

$$\left(x+2+\sqrt{3}\right)\left(x+2-\sqrt{3}\right)(=0)$$

THEN

$x = -2 \pm \sqrt{3}$ ***A1A1***

Note: Award ***A1*** for $-2+\sqrt{3}$ and ***A1*** for $-2-\sqrt{3}$.

[3 marks]

continued…

Question 2 continued.

(ii) **METHOD 1**

attempts to find their $\frac{1}{-2-\sqrt{3}} \times \frac{-2+\sqrt{3}}{-2+\sqrt{3}}$ ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$= \frac{-2+\sqrt{3}}{(-2)^2 - \left(\sqrt{3}\right)^2}$ and $(-2)^2 - \left(\sqrt{3}\right)^2 = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

Note: Award as above for attempting to show that $\frac{1}{-2+\sqrt{3}} = -2-\sqrt{3}$.

METHOD 2

considers the product of their roots ***(M1)***

Note: Only award ***(M1)*** if the roots are in the form $s \pm \sqrt{t}$.

$\left(-2+\sqrt{3}\right)\left(-2-\sqrt{3}\right) = 1$ ***A1***

hence the roots form a reciprocal pair ***AG***

METHOD 3

considers $\frac{c}{a}\left(=\frac{1}{1}\right)$ ***(M1)***

the product of roots is 1 ***A1***

hence the roots form a reciprocal pair ***AG***

[2 marks]

continued…

Question 2 continued.

(b) roots are $\pm\mathrm{i}$ $\left(=\mathrm{i},\frac{1}{\mathrm{i}},=\operatorname{cis}\left(\pm\frac{\pi}{2}\right),=\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}}\right)$ ***(A1)***

EITHER

$\frac{1}{-\mathrm{i}}\times\frac{\mathrm{i}}{\mathrm{i}}=\mathrm{i}$ OR $\frac{1}{-\mathrm{i}}=\mathrm{i}$ ***A1***

Note: Award as above for showing that $\frac{1}{\mathrm{i}}=-\mathrm{i}$ OR stating that $\frac{1}{\mathrm{i}}=-\mathrm{i}$.

OR

$\mathrm{i}\times-\mathrm{i}=1$ OR $\mathrm{i}\times\frac{1}{\mathrm{i}}=1$ ***A1***

OR

the product of roots is 1 ***A1***

THEN

hence the roots form a reciprocal pair ***AG***

[2 marks]

(c) $p\left(\frac{1}{x}\right)=\frac{a}{x^2}+\frac{b}{x}+a$ ***(A1)***

$x^2p\left(\frac{1}{x}\right)=x^2\left(\frac{a}{x^2}+\frac{b}{x}+a\right)$ ***A1***

$=ax^2+bx+a$

$=p(x)$ ***AG***

Note: Award as above for showing that $\frac{p(x)}{x^2}=p\left(\frac{1}{x}\right)=\frac{a}{x^2}+\frac{b}{x}+a$.

[2 marks]

continued…

Question 2 continued.

(d) **METHOD 1**

$p(\alpha) = 0 \Rightarrow \alpha^n p\left(\frac{1}{\alpha}\right) = 0$ ***A1***

EITHER

$\alpha^n \neq 0$ (as $\alpha \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

OR

$\alpha \neq 0$ (hence $\alpha^n \neq 0$) $\Rightarrow p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

THEN

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

METHOD 2

$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p\left(\frac{1}{\frac{1}{\alpha}}\right)$

$p\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n p(\alpha)$ ***A1***

for $\alpha \neq 0$ (seen anywhere), $p(\alpha) = 0 \Rightarrow \left(\frac{1}{\alpha}\right)^n p(\alpha) = 0$ and so $p\left(\frac{1}{\alpha}\right) = 0$ ***R1***

so $\frac{1}{\alpha}$ is also a root ***AG***

Note: The ***R1*** is dependent on the ***A1***.

[2 marks]

continued…

Question 2 continued.

(e) **METHOD 1**

$p(x) = x^n p\left(\frac{1}{x}\right)$

$q(x) = x^m q\left(\frac{1}{x}\right)$ ***(A1)***

attempts to substitute for $p(x)$ and $q(x)$ into $f(x) = p(x)q(x)$ ***(M1)***

$f(x) = x^n p\left(\frac{1}{x}\right) x^m q\left(\frac{1}{x}\right) \; \left(= x^{n+m} p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right)\right)$ ***A1***

$= x^{n+m} f\left(\frac{1}{x}\right)$ (and $n+m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

METHOD 2

$p\left(\frac{1}{x}\right) = \frac{p(x)}{x^n}$

$q\left(\frac{1}{x}\right) = \frac{q(x)}{x^m}$ ***(A1)***

attempts to substitute for $p\left(\frac{1}{x}\right)$ and $q\left(\frac{1}{x}\right)$ into $f\left(\frac{1}{x}\right) = p\left(\frac{1}{x}\right) q\left(\frac{1}{x}\right)$ ***(M1)***

$f\left(\frac{1}{x}\right) = \frac{p(x)}{x^n} \frac{q(x)}{x^m} \; \left(= \frac{p(x)q(x)}{x^{n+m}}\right)$ ***A1***

$x^{n+m} f\left(\frac{1}{x}\right) = p(x)q(x)$ and $f(x) = p(x)q(x)$ (and $n+m$ is the degree of f) ***A1***

hence f is a palindromic polynomial. ***AG***

[4 marks]

continued…

Question 2 continued.

METHOD 3

for example, $p(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_{n-1}x+a_n$ AND

$q(x)=b_mx^m+b_{m-1}x^{m-1}+...+b_{m-1}x+b_m$ ***A1***

attempts to multiply out their $p(x)q(x)$ ***(M1)***

Note: Only award ***(M1)*** for polynomials of degree n and m respectively with different corresponding coefficients.

$p(x)q(x)=a_nb_mx^{m+n}+(a_{n-1}b_m+a_nb_{m-1})x^{m+n-1}+...+(a_{n-1}b_m+a_nb_{m-1})x+a_nb_m$

A1A1

Note: Award ***A1*** for the first and second terms and ***A1*** for the penultimate and last terms.

hence f is a palindromic polynomial ***AG***

[4 marks]

(f) **METHOD 1**

attempts to expand the product of two quadratic factors ***(M1)***

EITHER

$x^4+ux^3+vx^3+uvx^2+2x^2+ux+vx+1$ ***(A1)***

OR

$x^4+(u+v)x^3+(uv+2)x^2+(u+v)x+1$ ***(A1)***

THEN

equates coefficients and forms a system of two equations in u and v ***(M1)***

$u+v=2$ and $uv+2=-1$ $(uv=-3)$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u=-1, v=3$ OR $(f(x)=)(x^2-x+1)(x^2+3x+1)$ ***A1***

Note: Candidates must attempt to form and solve a system of equations in u and v. Hence, award no marks for working backwards from either decimal or exact values of roots given by a GDC. Award ***A0*** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

continued…

Question 2 continued.

METHOD 2

attempts to divide one of $\left(x^2+ux+1\right)$ or $\left(x^2+vx+1\right)$ into $f\left(x\right)$ ***(M1)***

EITHER

for example, the quotient is $x^2+\left(2-u\right)x+u^2-2u-2$ ***A1***

sets their quotient equal to x^2+vx+1 ***(M1)***

$x^2+\left(2-u\right)x+u^2-2u-2=x^2+vx+1$

$u^2-2u-2=1$ and $2-u=v$ ***A1***

attempts to solve their equations for u and then v (or vice versa) ***(M1)***

OR

for example, the remainder is $u\left(-u^2+2u+3\right)x+\left(-u^2+2u+3\right)$ ***A1***

sets their remainder equal to zero ***(M1)***

attempts to solve for u (or v) ***(M1)***

$u=-1,3$ and $u+v=2$ ***A1***

THEN

$u=-1, v=3$ OR $\left(f\left(x\right)=\right)\left(x^2-x+1\right)\left(x^2+3x+1\right)$ ***A1***

Note: Award ***A0*** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

continued…

Question 2 continued.

METHOD 3

considers $f(1)$, for example ***(M1)***

$(u+2)(v+2)=5$ ***A1***

considers $f(2)$, for example ***(M1)***

$(2u+5)(2v+5)=33$ ***A1***

attempts to solve their system of two equations ***(M1)***

$u=-1, v=3$ OR $(f(x)=)(x^2-x+1)(x^2+3x+1)$ ***A1***

Note: Award ***A0*** for $u=-1, v=3$ and $u=3, v=-1$ both specified.

[6 marks]

(g) attempts to solve $x^4+2x^3-x^2+2x+1=0$ ***(M1)***

Note: Award ***(M1)*** for attempting to solve one of their quadratic equations.

$\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}, -\frac{3}{2}\pm\frac{\sqrt{5}}{2}$ ***A1A1***

Note: Award ***A1*** for $\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$ and ***A1*** for $-\frac{3}{2}\pm\frac{\sqrt{5}}{2}$.

[3 marks]

continued…

Question 2 continued.

(h) **METHOD 1**

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_2x^2 + a_1x + a_0\ (= 0)$

attempts to find $p(-1)$ ***(M1)***

$p(-1) = (-1)^n + a_1(-1)^{n-1} + a_2(-1)^{n-2} + ... + a_2(-1)^2 + a_1(-1) + 1$ ***(A1)***

$= (-1) + a_1 - a_2 + ... + a_2 - a_1 + 1$ ***A1***

(since n is odd), $p(x)$ has

EITHER

$n+1$ terms so all terms cancel giving $p(-1) = 0$ ***R1***

OR

an even number of terms so all terms cancel giving $p(-1) = 0$ ***R1***

THEN

hence -1 is always a root ***AG***

METHOD 2

considers the product of roots of a polynomial equation of degree n

(M1)

$$\frac{(-1)^n a_0}{a_n}$$

product of roots is $(-1)^n$ (seen anywhere) ***A1***

(for n odd) there are $\frac{n-1}{2}$ reciprocal pairs of roots ***A1***

product of each reciprocal pair is 1 and $(-1)^n = -1$ (since n is odd) ***R1***

hence -1 is always a root ***AG***

continued…

Question 2 continued.

METHOD 3

consider $p(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_2x^2 + a_1x + a_0\ (= 0)$

considers $p(x) = x^n p\left(\frac{1}{x}\right)$ with $x = -1$ ***(M1)***

$p(-1) = (-1)^n p(-1)$ ***A1***

$p(-1) = -p(-1)\ \ (2p(-1) = 0)$ (since n is odd) ***A1***

hence $p(-1) = 0$ ***R1***

hence -1 is always a root ***AG***

[4 marks]

Total [28 marks]